Fixing by thinking: The power of dimensional analysis

E. N. Economou

Department of Physics, University of Crete and
Foundation for Research and Technology-Hellas (FORTH), Heraklio, Crete, Greece

Abstract. The basic ideas of science are: the atomic one, the minimization of free energy, and the particle/wave duality; the latter can be distilled into three fundamental principles of Quantum Mechanics: Heisenberg, Pauli, Schrödinger principles. The universal constants $\hbar$, $c$, and the ones associated with elementary particles and forces together with the basic laws of nature outlined above and combined with dimensional analysis allows us to obtain an "enormous amount of information" (to quote Feynman) concerning light-matter interactions, mass of nucleons, and the structure of nuclei, atoms, molecules, solids, liquids, planets, stars, dead stars, black holes, and finally, the history of the Universe.

INTRODUCTION

In the atomic idea there is an enormous amount of information about the world, if just a little imagination and thinking are applied.

R.P. Feynman

Three basic ideas constitute the foundation of modern physical sciences:

1. The atomic idea, according to which everything is made of indestructible, indivisible, microscopic particles which attract each other to form composite structures.

To substantiate this idea one has to know: (a) which are the properties of these elementary particles; (b) what kind of interactions lead to their attraction, (c) what counterbalances this attraction and establishes equilibrium.

In tables 1 and 2, I present the current answers to (a) and (b) including the numbers which characterize the elementary particles and their interactions; besides these numbers, we need also two truly universal constants, the $\hbar = 10^{-34}$ J$\times$s (trademark of Quantum Mechanics) and $c = 3 \times 10^8$ m/s, the velocity of light in vacuum.

TABLE 1. Leptons and quarks (coming in three families) are the established elementary particles [1]. Only three of them ($e, u, d$) are participating in the structure of ordinary matter. Neutrinos are too light and weakly interacting to be bound in ordinary matter. The search for the Higgs particle(s) is under way, while more exotic particles may account for the dark matter and, possibly, for the dark energy.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Mass (MeV)</th>
<th>Charge</th>
<th>Spin</th>
<th>Symbol</th>
<th>Mass (MeV)</th>
<th>Charge</th>
<th>Spin</th>
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</thead>
<tbody>
<tr>
<td>$e$</td>
<td>0.51</td>
<td>-1</td>
<td>1/2</td>
<td>u</td>
<td>1.5-4.5</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\nu_e$</td>
<td>$-10^4$?</td>
<td>0</td>
<td>1/2</td>
<td>d</td>
<td>5-8.5</td>
<td>-1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\mu$</td>
<td>105.7</td>
<td>0</td>
<td>1/2</td>
<td>c</td>
<td>1000-1400</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\nu_\mu$</td>
<td>$-10^4$?</td>
<td>0</td>
<td>1/2</td>
<td>s</td>
<td>80-155</td>
<td>-1/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1777</td>
<td>-1</td>
<td>1/2</td>
<td>t</td>
<td>~174000</td>
<td>2/3</td>
<td>1/2</td>
</tr>
<tr>
<td>$\nu_\tau$</td>
<td>$-10^7$?</td>
<td>0</td>
<td>1/2</td>
<td>b</td>
<td>4000-4500</td>
<td>-1/3</td>
<td>1/2</td>
</tr>
</tbody>
</table>
### TABLE 2. Properties of the four basic interactions[1].

<table>
<thead>
<tr>
<th>Name</th>
<th>Dimensionless Strength</th>
<th>Range</th>
<th>Type</th>
<th>Carrier Particle</th>
<th>Emitters/Acceptors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational</td>
<td>( \alpha_g = \frac{Gm_c^2}{\hbar c} = 5.9 \times 10^{-39} )</td>
<td>( \infty )</td>
<td>Attractive</td>
<td>Graviton</td>
<td>All particles</td>
</tr>
<tr>
<td>Electromagnetic</td>
<td>( \alpha_e = \frac{e^2}{\hbar c} = \frac{1}{137} )</td>
<td>( \infty )</td>
<td>Attractive or repulsive</td>
<td>Photon</td>
<td>Charged particles</td>
</tr>
<tr>
<td>Weak “nuclear”</td>
<td>( \alpha_w = \frac{g^2}{\hbar c} \sqrt{\frac{2m_r}{m_w}} = 10^{-8} )</td>
<td>( 10^{-15} )</td>
<td>( u \subseteq d + ... )</td>
<td>Vector bosons</td>
<td>All “matter” particles</td>
</tr>
<tr>
<td>Strong “nuclear”</td>
<td>( \alpha_s = \frac{g^2}{\hbar c} = 15 )</td>
<td>( 10^{-15} )</td>
<td>Attractive or repulsive</td>
<td>gluons</td>
<td>Quarks and gluons</td>
</tr>
</tbody>
</table>

### LAWS OF MOTION

The answers to the question “what counterbalances the overall attraction and establishes equilibrium” constitute the second and the third basic ideas:

2. **The wave-particle duality** (i.e. Quantum Mechanics) rescues the structures of matter from collapse by providing the kinetic energy which counterbalances the squeezing action of the interactions. The content of QM can be distilled in three basic principles:

- **Heisenberg principle:** \( \Delta x \Delta p \geq \frac{\hbar}{2} \). From this principle the following inequalities follow

\[
\langle \varepsilon_k \rangle \geq \frac{\Delta p^2}{2m} \geq \frac{9}{8} \frac{\hbar^2}{m \Delta x^2} \approx 4.87 \frac{\hbar^2}{m V^{2/3}},
\]

assuming that the relation between kinetic energy \( \varepsilon_k \) and momentum \( p \) is the non-relativistic one, \( \varepsilon_k = p^2 / 2m \), where \( m \) is the mass of the particle. In the extreme relativistic limit, where \( \varepsilon_k = cp \), the Heisenberg principle leads to the following inequality:

\[
\langle \varepsilon_k \rangle \geq c |\Delta p| \geq \frac{2 \hbar c}{3} \frac{1}{|\Delta x|} \approx 3.12 \frac{\hbar c}{V^{1/3}}.
\]

Notice that the volume \( V \) (assumed spherical) within which the particle is confined is in the denominator as \( V^{1/3} \approx r^{-2} \) (in the non-relativistic case) or as \( V^{1/3} \approx r^{-1} \) (in the extreme relativistic case); these exponents are of critical importance: if the interaction is of the form \( -A/r \), a kinetic energy of the form \( B/r^2 \) would dominate for small size \( r \) and, as a result, it would not allow the structure to collapse; on the other hand, if the kinetic energy is of the form \( C/r \) (as in the extreme relativistic case) the possibility of collapse cannot be ruled out (Actually, it does happen, if \( A > C \); recall the case of a neutron star collapsing to a black hole).

The second basic principle of QM is the Pauli principle, which, for identical particles of spin \( 1/2 \) sharing the same volume, can be expressed as follows:

- **Pauli exclusion principle:** No more than two identical spin \( 1/2 \) particles can be in the same spatial state.

  If we have \( N \) identical, spin \( 1/2 \), particles sharing the volume \( V \), one way to satisfy Pauli principle is to divide the volume \( V \) into \( N/2 \) equal subvolumes each one of volume \( 2V/N \) and to place only one pair (spin up/spin down) in each subvolume. This is not the most efficient way to satisfy the Pauli principle, but it produces essentially the same result as far as the total average kinetic energy is concerned. By replacing \( V \) by \( 2V/N \) is Eqs (1a) and (1b) and adjusting the numerical factors, we obtain
\begin{align}
\langle E_k \rangle & \equiv \sum_{i=1}^{N} \langle E_{ik} \rangle \geq N2.87 \frac{\hbar^2 N^{2/3}}{m V^{2/3}}, \quad (2a) \\
\langle E_k \rangle & \equiv \sum_{i=1}^{N} \langle E_{ik} \rangle \geq N2.32 \frac{\hbar c N^{1/3}}{V^{2/3}}. \quad (2b)
\end{align}

Thus Pauli’s principle leads to a tremendous increase in the minimum kinetic energy of a system of \( N \) identical spin \( 1/2 \) particles sharing the same volume \( V \).

The third basic principles of QM, which I shall call Schrödinger’s principle, states that the first excited level of a particle of mass \( m \) confined in a volume \( V \) is higher than its ground state energy by a non-zero amount \( \delta \epsilon \) given, is the non-relativistic case, by

\[
\delta \epsilon \equiv \epsilon_1 - \epsilon_2 = c_1 \frac{\hbar^2}{m V^{2/3}},
\]

where \( c_1 \) is a numerical constant of order unity.

One can hardly overestimate the importance of these three principles for the structure of all kind of matter. Eq. (2a) secures the existence of the World as we know it by preventing the collapse of all matter to one or more black holes and by establishing equilibrium when the squeezing pressure, \( P_{ext} \), of the interactions becomes equal to the expanding pressure, \( P_e \), of the quantum kinetic energy. Eq. (3) makes sure that this equilibrium is stable even in the presence of temporary external finite perturbations as long as \( \epsilon_{pert} \) is smaller than \( \delta \epsilon \); this, e.g., explains the admirably predictable physicochemical behavior of atoms in various environments; on the other hand, this stability is not absolute as to exclude changes; on the contrary, the possibility of changes is incorporated and it is realized when \( \epsilon_{pert} > \delta \epsilon \).

The third basic idea quantifies the equilibrium condition by generalizing the equation \( P_{ext} = P_e \) as follows

3. **Equilibrium corresponds to the minimization of the (free) energy.**

Under the common conditions of constant external pressure, \( P_{ext} \), and constant external temperature, \( T_{ext} \), the equilibrium corresponds to the minimum of the Gibbs free energy \( G \), where

\[
G \equiv E_t + P_{ext} V - T_{ext} S,
\]

\( S \) is the entropy of the system, and \( E_t + P_{ext} V \) is the enthalpy \( H \). When \( P_{ext}, T_{ext} \) are negligible, the equilibrium corresponds to the minimization of the total energy \( E_t \).

**INTERACTIONS OF ELECTROMAGNETIC (EM) FIELDS WITH MATTER**

**Photons in equilibrium**

Consider first photons being confined in a large cavity of volume \( V(V \rightarrow \infty) \) bounded by matter of temperature \( T \). Photons come to their equilibrium state through their interaction (absorption and emission) with the surrounding matter. We would like to obtain the energy density of equilibrated photons at temperature \( T \). Obviously this energy density would depend on \( T \) (through the combination \( k_e T \), where \( k_e \) is Boltzmann’s constant; in nature \( T \) always appear through the product \( k_e T \)); it would depend also on \( c \) and \( h \) since \( c \) and \( h \) are the only universal constants

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\(^{1}\) By a factor of the order \( N^{2/3} \) in the non-relativistic case, and \( N^{1/3} \) in the extreme relativistic case.
characterizing a photon. Out of the three quantities \( B k T \), \( c \) and \( \hbar \) there is a unique combination having dimensions of energy over volume (\( k_B T \) is energy, \( c \) is length over time, and \( \hbar \) is energy \times\) time; hence \( \hbar / k_B T \) has dimension of time, \( hc / k_B T \) has dimensions of length, and \( k_B T / (hc / k_B T)^3 \) has dimensions of energy over volume). Thus

\[
\text{photon energy density} = c_1 \frac{(k_B T)^4}{\hbar^3 c^3},
\]

where \( c_1 \) is a numerical constant of the order of one (actually \( c_1 = \pi^2 / 15 \)). From (5) we have for the photon energy \( U_{ph} \)

\[
U_{ph} = \frac{\pi^2 V (k_B T)^4}{15 \hbar^3 c^3}.
\]

The rest of the thermodynamic quantities for a photon gas in equilibrium can be obtained from the relations

\[
dU = TdS, \quad F = U - TS, \quad P = -\frac{\partial F}{\partial V}, \quad C = T \frac{\partial S}{\partial T}.
\]

The results are

\[
S_{ph} = \frac{4 U_{ph}}{3 T},
\]

\[
F_{ph} = -\frac{1}{3} U_{ph},
\]

\[
P = \frac{1}{3} \frac{U_{ph}}{V},
\]

\[
C_{ph} = \frac{4 U_{ph}}{T}.
\]

A black body is one which absorbs all the incoming EM energy flux (no reflection). Such a body being at temperature \( T \) must emit EM radiation \( J_o \) per unit surface and per unit time equal to the one it receives, per unit surface and per unit time, from a photon gas at temperature \( T \) in contact and in equilibrium with the black body. \( J_o \) can be calculated by integrating the normal component, \( j_\perp \) of the received energy flux, \( j_\perp = c(U / V) \cos \theta / 4\pi \), over half the solid angle: \( J_o = \int_0^{\pi/2} \int_0^{\pi/2} d\phi d\theta \sin \theta j_\perp \). The result is

\[
J_o = \frac{1}{4} \frac{c U}{V} = \frac{\pi^2}{60} \frac{(k_B T)^4}{c^2 \hbar^3} \equiv \sigma T^4.
\]

\( J_o \) by definition has dimension of energy per unit time and per unit surface; hence, it ought to be proportional to \( (k_B T)^4 / c^2 \hbar^3 \).

**Emission of radiation by an accelerating charged particle or by an oscillating dipole**

For a particle of charge \( q \) and acceleration \( a(t) \) the EM radiated energy per unit time, \( I \), depends on \( q \) (if \( q = 0 \), there is no radiation), on \( a \) (if \( a = 0 \), there is no radiation) and, of course, on \( c \) since \( c \) is the only universal
constant related to EM radiation. There is only one combination of $q$, $a$, and $c$ with dimensions of energy per unit time

$$I(t) = c_a^2 q^2 / c^3.$$  \hfill (12)

The numerical constant $c_1$ is equal to $2/3$, if the time dependence of $a$ is sinusoidal and $a^2$ is replaced by the time average $\overline{a^2}$.

For an oscillating dipole, $I$ depends on the dipole moment $p$, on the frequency $\omega$ (if $\omega = 0$, there is no radiation), and of course, on $c$. Thus

$$I = c_1 p^2 \omega^4 / c^3 = 2 \frac{p^2 \omega^4}{c^3}.$$  \hfill (13)

Eq. (13) can be derived also from Eq. (12) by taking into account that $p = q \cdot r$.

Eq (13) allows us to obtain an estimate for the lifetime $\tau$ of an excited atomic state which is deexcited by emitting a photon through dipole interaction. Indeed, the total energy emitted during the lifetime is $I\tau$ which must be equal to the energy $\hbar \omega$ of the emitted photon. Hence, according to (13)

$$\tau = \frac{1}{c_1} \frac{\hbar c^3}{(e \cdot r)^2 \omega^3},$$  \hfill (14)

where $c_1$ is a numerical factor of the order of one. For the transition $2p \rightarrow 1s$ in the hydrogen atom we can estimate $\tau$ by taking $r = a_p$, $\hbar \omega = e_2 - e_1 = 10.2 \text{ eV}$, and $c_1 = 1$. We find

$$\tau \approx 1.18 \times 10^{-9} \text{ s}$$  \hfill (15)

vs. $\tau = 1.59 \times 10^{-9} \text{ s}$ obtained by a detailed quantum mechanical calculation.

**Scattering of a photon by a charged particle or a neutral atom or molecule**

The scattering cross-section $\sigma$ of such an event is defined as the ratio $\Gamma / F$, where $\Gamma$ is the number of scattered photons per unit time and $F$ is the flux of the incoming photons:

$$\Gamma = I / \hbar \omega$$

and

$$F = \text{concentration of photons} \times c = c \frac{E_o^2}{4\pi \hbar \omega}.$$  

Thus

$$\sigma = \frac{I}{\hbar \omega} \frac{4\pi \hbar \omega}{cE_o^2} = \frac{4\pi I}{cE_o^2}.$$  \hfill (16)

For a charged particle $I = \frac{2}{3} q^3 a^2 / c^3$ and $a = qE_o / m$. Thus
\[ \sigma = \frac{8\pi}{3} \left( \frac{q^2}{mc^2} \right)^2 = 6.65 \times 10^{-29} \text{m}^2, \] (17a)

where \( q^2 / mc^2 = r_e = 2.82 \times 10^{-13} \text{m} \) is the classical “radius” of the electron (resulting by equating the rest energy \( mc^2 \) to the electrostatic energy \( q^2 / r_e \)). Eq. (17a) can be obtained by dimensional analysis taking into account that the scattering cross-section \( \sigma \) must depend on \( q \) (for \( q = 0 \) there is no scattering), on \( m \) (for \( m = \infty \) there is no motion of the particle and, therefore, no scattered radiation), on \( c \), and on \( \hbar \omega \). Thus, dimensional analysis predicts for \( \sigma \) the following expression (known as the Klein-Nishina formula)

\[ \sigma = c_i \left( \frac{q^2}{mc^2} \right)^2 f \left( \frac{\hbar \omega}{mc^2} \right), \] (17b)

where the function \( f \) of the dimensionless ratio \( \hbar \omega / mc^2 \) cannot be determined by dimensional analysis. Actually, if \( c_i \) is taken equal to \( 8\pi / 3 \), \( f \) is a decreasing function of \( \hbar \omega / mc^2 \) which is one for \( \hbar \omega = 0 \) and zero for \( \hbar \omega / mc^2 \to \infty \).

The scattering cross-section of a photon by a neutral atom or molecule of zero average dipole moment \( p \) is due to the induced dipole moment \( p \) by the external field \( E_o \)

\[ p = a_p E_o, \] (18)

where \( a_p \) is the polarizability of the atom or the molecule. Substituting Eqs (13) and (18) in (16) we obtain

\[ \sigma = \frac{8\pi}{3} \omega^4 \rho^3 \sigma_p^2. \] (19)

The polarizability \( a_p \) has dimensions of length cube

\[ a_p = c_i r_o^3 f_R(\omega), \] (20)

where \( c_i \) is a numerical factor of the order of one, \( r_o \) is the radius of the atom (or any other length characterizing the size of molecule) and \( f_R(\omega) \) is a typical resonance function

\[ f_R(\omega) = \sum_i \frac{\omega_{oi}^2}{\omega_{oi}^2 - \omega^2 - i\omega / \tau_i}, \]

which takes into account the strongly increased response of the atom (or molecule) when the frequency \( \omega \) of the incoming EM field coincides with one of its eigenfrequencies \( \omega_{oi} = |e_{oi} - e_o| / \hbar \). For \( \omega \ll \omega_{oi} \), \( f_R(\omega) = 1 \), while for \( \omega = \omega_{oi} \)

\[ |f_R(\omega)| = \omega_{oi} \tau_i, \]

where \( \tau_i \) is the lifetime given by Eq (14). Thus, for \( \omega \ll \omega_{oi} \) we have that
\[ \sigma = 3 \times 10^5 \frac{r_0^6}{\lambda^4}, \quad \omega \ll \omega_{\text{al}}, \quad (21) \]

while, for \( \omega = \omega_{\text{al}} \) we obtain

\[ \sigma = c^2 3.6 \times 10^6 r_0^2, \quad \omega = \omega_{\text{al}}. \quad (22) \]

Eq. (21) means that the off-resonance scattering cross-section of an optical photon by a neutral atom or molecule is of the order of \( 10^{-30} \) or \( 10^{-29} \text{m}^2 \), i.e., about ten orders of magnitude less than the geometrical cross-section \( \pi r_0^2 \).

On the other hand, at resonance the scattering cross-section is six orders of magnitude larger than the geometrical cross-section!

**FORM QUARKS TO THE UNIVERSE**

The rest energy of a proton or a neutron

We shall show that these rest energies which are \( m_e c^2 = 938.27 \text{MeV} \) for the proton and \( m_n c^2 = 939.57 \text{MeV} \) for the neutron are mainly due to the quantum kinetic energy of the confined quarks. The rest energies of the latter, as shown in table 1, are less than 1% of \( m_e c^2 \); hence Eq. (1b) is the appropriate one. Out of the nine degrees of freedom corresponding to the three quarks making up the proton (or neutron), three are associated with its center of mass; the center of mass is not confined and, hence, its minimum kinetic energy is zero. Only six degrees of freedom corresponding to two particles are confined within the volume \( V = 4 \pi r^3 / 3 \), where \( r_p = 0.8 \text{fm} \) is the radius of the proton or the neutron. Thus

\[ m_e c^2 = m_n c^2 = \frac{3}{2} \frac{12 \hbar c}{\nu^{1/3}} \approx 953.8 \text{MeV}, \quad (23) \]

i.e., very close to the actual values (relative error 1.5%).

The energy of a nucleus with \( A = Z + N \) nucleons

The energy of a nucleus consisting of \( Z \) protons and \( N \) neutrons is equal to \( Z m_p c^2 + N m_n c^2 - B \) where \( B \) is the binding energy of all the nucleons. Three are the main contributions to \( -B \):

1. The strong nuclear interaction gives a contribution equal to \( N_{\text{pairs}} \mathcal{V} \) where \( N_{\text{pairs}} \) is the number of interacting pairs of nucleons and \( \mathcal{V} \) is the average interaction energy per interacting nucleon pair (\( \mathcal{V} = -11.3 \text{MeV} \)). Because of the short range character of the strong nuclear force only nearest neighbor pairs interact. Thus

\[ N_{\text{pairs}} = \frac{1}{2} A_g N_g + \frac{1}{2} A_s N_s, \]

where \( A_g \) and \( A_s \) are the number of nucleons in the bulk (i.e., in the interior of the nucleus) and in the surface respectively. \( A_g \sim R^2 \sim A_s^{2/3} \) and \( A_g = A - A_s \), where \( R \sim A_s^{1/3} \) is the radius of the nucleus. \( N_g = 8 \) is the average number of nearest neighbors for a nucleon in the bulk and \( N_s = 5 \) for a nucleon in the surface. We have then by choosing \( A_s = A_s^{2/3} \) the following result
\[ E_s = -45.2A + 17A^{2/3} \text{ in MeV}. \] (24)

2. The second contribution to \(-B\) is due to the kinetic energy of the nucleons; it is repulsive, and it is given (according to (2a)) by

\[ E_k = 2.87 \frac{h^2}{mV^{2/3}} (Z^{5/3} + N^{5/3}). \]

Taking into account that \(V = 4\pi R^3 / 3\), \(R = 1.2A^{1/3}\) (in fm) and

\[ Z^{5/3} + N^{5/3} = \frac{1}{2^{2/3}} \left[ A^{5/3} + \frac{(N-Z)^2}{A^{1/3}} \right] \]

we can rewrite \(E_{kin}\) as follows

\[ E_{kin} = 30.6A + 17\frac{(N-Z)^2}{A} \text{ in MeV}. \] (25)

3. The Coulomb repulsion among protons is

\[ E_c = \frac{1}{2} \sum_{i,j=1}^{Z} \frac{e^2}{r_{ij}} = \frac{1}{2} \frac{Z(Z-1)e^2}{r}, \quad r = \frac{5}{6}R \]

or

\[ E_c = 0.71\frac{Z(Z-1)}{A^{1/3}}, \text{ in MeV}. \] (26)

From (25) we see that the kinetic energy favors \(N = Z = A/2\), while the Coulomb energy favors \(N = A\). For small nuclei the kinetic energy dominates over the Coulomb energy so that \(N = Z\). As the size \(A\) increases the Coulomb energy grows faster than the kinetic one, and, as a result, the balance tips to more neutrons than protons. The ratio \(Z/A\) can be found by minimizing the total energy (sum of Eqs. (24), (25), and (26)) with respect to \(Z\) under \(A = \text{const}\). We obtain

\[ \frac{Z}{A} \approx \frac{1 + 0.0075/A^{1/3}}{2 + 0.015A^{2/3}}. \] (27)

Notice that weak interactions allow nucleus to change neutrons to protons and vice versa until the optimum ratio \(Z/A\) (given by Eq. (27)) is achieved. By substituting Eq. (27) in \(-B = E_s + E_k + E_c\) and by dividing by \(A\) we obtain the quantity \(-B/A\) as a function of \(A\) (see Fig.1).

Eqs. (24), (25), (26), and (27) together allow \(-B = E_s + E_{kin} + E_c \pm \delta\), where \(\pm \delta\) is a small correction favoring even/even nuclei (\(-\delta\) is for even/even nuclei and \(+\delta\) is for odd/odd nuclei) allows us to explain several important phenomena (e.g., why the products of fission are radioactive emitting electrons as radiation \(\beta\)? Why Uranium 235 is fissible, while Uranium 238 is not? etc).
FIGURE 1. The energy reduction per nucleon, $-B/A$ (as a result of the nucleus formation) vs. the number of nucleons $A$. The rise of the curve for small $A$ is due to the increased number of nucleons at the surface (where the exploitation of strong nuclear force is less effective). The rise for large $A$ is due to the repulsive Coulomb forces. This last rise makes possible the extraction of energy by fission (i.e., by splitting the initial large nucleus to two smaller ones); it is also responsible for the non-existence of nuclei with $A$ larger than about 240 (because then the splitting is (almost) spontaneous).

Atoms

The structure of matter from the level of atoms to the level of an asteroid (of linear dimension less than a few hundred kilometers) is dominated by only one of the four forces: the electromagnetic one; its strength is determined by the charge $e$ (of the proton). The quantum kinetic energy, which counterbalances this force depends (according to Eq. (1a)) on the combination $\hbar^2/m$. The dominant kinetic energy will be the one with the smaller mass in the denominator, i.e., the mass of the electron, $m_e$. The three quantities $e$, $\hbar$, $m_e$ define a system of units. E.g.,

- unit of length: $a_B = \hbar^2/m_e e^2 = 0.529 \text{Å}$
- unit of energy: $E_o = \hbar^2/m_e a_B^2 = e^2/a_B = 27.2 \text{eV}$
- unit of time: $t_o = \hbar/e_o = 2.42 \times 10^{-17} \text{s}$
- unit of velocity: $v_o = a_B/t_o = h/m_e a_B = e^2/\hbar = c/137$
- unit of pressure: $P_o = e_o/a_B^3 = \hbar^2/m_e a_B^5 = 2.94 \times 10^8 \text{bar}$
- unit of temperature: $T_o = e_o/k_B = 27.2 \times 11600 = 3.16 \times 10^5 \text{K}$
- unit of density: $\rho_{iso} = m_e/a_B^3 = 6.15 \text{kg/m}^3$
- unit of electrical resistance: $R_o = \hbar/e^2 = 4108 \text{ohm}$
- unit of resistivity: $\rho_o = R_o a_B = 21.7 \mu\text{ohm·cm}$

The most important quantities of an atom is its size characterized by its radius $a_r$ (as determined by the position of the maximum of the highest occupied orbital) and its ionization potential $I_o$. Both of them depend mainly on $e$, $\hbar$, $m_e$; the mass of the nucleus $m_o$ would enter through the reduced mass $m_o m_e/(m_o + m_e) = m_e [1 + (m_e/m_o)]$ as a very small correction less than $1\%$. The velocity of light would not influence appreciably the highest occupied orbital; similarly the external pressure and the external temperature do not play any significant role unless their values are comparable to the $P_o$ and $T_o$. Thus we conclude that
\[ r_a = c_a a_B, \quad (28) \]

\[ I_a = c_I \frac{e^2}{a_B} = c_I \frac{e^2}{r_a}. \quad (29) \]

The numerical constant \( c_a \) depends on the particular atom and its value is in the range from about one, to about five, with an average value over the periodic table equal to 2.6; the local maximum values are for alkali atoms (\( c_a = 3, 3.24, 4.09, 4.32, 4.76 \) for \( Li, Na, K, Rb, Cs \) respectively) and the local minimum values for noble gas atoms (\( c_a = 0.55, 0.669, 1.25, 1.50, 1.86, 2.06 \) for \( He, Ne, Ar, Kr, Xe \) and \( Rn \) respectively). The numerical constant \( c_I \) in Eq. (29) is in a narrower range (between about 0.45 and 0.75) than \( c_I' \); thus it is preferable to replace \( a_B \) by the actual radius \( r_a \) of each atom.

We need also to have some information regarding the atomic eigenstates, the so called atomic orbitals. The radial part of each orbital is characterized by the main quantum number \( n \) such that the extend of the orbital is roughly proportional to \( n^2 \); the angular part is the same as the angular part of polynomials of degree \( \ell \) (all terms being of degree \( \ell \) which satisfy Laplace equation; the \( \ell = 0 \) orbitals, denoted as \( s \) orbitals, do not depend on the angles; the \( \ell = 1 \) orbitals, denoted as \( p_z, p_y, p_x \) have the angular dependence of \( x/r, y/r \) and \( z/r \), and so on. The relative position of the energy levels which account for the structure of the periodic table can be obtained by using the results for the hydrogen and some simple physical arguments. The main results are the following:

\[ \varepsilon_{n,l} < \varepsilon_{n+1,l} \quad i = s, p, d, f, \cdots \quad \text{and} \quad \varepsilon_{ns} < \varepsilon_{np} < \varepsilon_{nd} < \varepsilon_{nf} \cdots, \quad (30) \]

\[ \varepsilon_{n+2,s} = \varepsilon_{n+1,d} = \varepsilon_{n,f} < \varepsilon_{n+2,p} < \varepsilon_{n+3,s} \quad (31) \]

\section*{Diatomic molecules}

The interaction energy of the two atoms at a distance \( d \) apart can be easily shown to be \(-A/d^6\), when \( d \gg r_{a1} + r_{a2} \), and \( Z_1 Z_2 e^2 / d \) as \( d \to 0 \). Thus there must be a negative minimum at a distance \( d_o \); this equilibrium distance must occur when the electronic clouds of the two atoms has a small degree of overlap. Thus

\[ d_o = c_1 (r_{a1} + r_{a2}), \quad (32) \]

where \( c_1 \) is a numerical factor usually a little less than one (strong bonds correspond to smaller \( c_1 \)). The binding energy \( \varepsilon_b \) of the molecule is necessarily of the order of the unit of energy \( \varepsilon_o = h^2 / 2 m a_o^2 \) with \( a_o \) replaced by \( d_o \).

\[ \varepsilon_b = c_2 \frac{h^2}{m d_o^2}, \quad (33) \]

where the numerical factor \( c_2 \) is usually less than one; it is smaller for single weak bonds and larger for strong triple bonds. The quantum of vibrational energy \( h \omega \) will be proportional to \( 1/\sqrt{\mu} \) as any harmonic vibration where \( \mu = m_{a1} m_{a2} / (m_{a1} + m_{a2}) \) is the reduced mass. Hence, it must be of the order of the unit of energy \( \varepsilon_a = e^2 / a_o \) multiplied by the dimensionless quantity \( \sqrt{m_a / \mu} \) and with \( a_o \) replaced by the bond length \( d_o \).
\[ \hbar \omega = c_3 \frac{e^2}{d_0} \sqrt{\frac{m_e}{\mu}}, \]  

(34)

where the numerical “constant” \( c_3 \) is of the order of one with larger values for strong triple bonds and smaller values for weak single bonds. (E.g., by taking \( c_3 = 1 \) we find 644 meV, 116 meV, 98 meV, 32 meV for \( \text{H}_2, \text{N}_2, \text{O}_2, \) and \( \text{Na}_2 \) respectively, while actual values are 546, 292, 196 and 19.7 meV respectively).

The rotational quantum \( \hbar^2 / I_r \), where \( I_r \) is the moment of inertia, \( I_r = \mu d_o^2 \) is directly related with the bond length

\[ \frac{\hbar^2}{I_r} = 27.2 \left( \frac{m_e}{\mu} \right) \left( \frac{a_{\text{B}}}{d_o} \right)^2 \text{eV} \]

(35)

(For \( \text{O}_2 \) this quantum is 0.358 meV \( \cong 4.16 \text{K} \)).

**Solids [3]**

In solids, as in molecules, the atoms touch each other so that their electronic clouds could have a small overlap; this means that the equilibrium nearest neighbor distance \( d_o \) is as in Eq. (32). Instead of \( d_o \) it is more convenient to use the average radius per atom \( r_c \) defined by the relation

\[ \frac{4\pi}{3} r_c^3 = \frac{V}{N_a}, \]

(36)

where \( V / N_a \) is the volume per atom which is the inverse of the concentration of atoms, \( n_a = N_a / V \). This radius \( r_c \) is directly related with the mass density, \( \rho_M \) (one of the main characteristics of condensed matter)

\[ \rho_M = \frac{m_a}{(4\pi/3)r_c^3} = 2.68 \frac{A_y}{f_c} \text{(in g/cm}^3 \text{)}, \]

(37)

where \( A_y \) is the (average) atomic weight and \( f_c \) is defined by the relation

\[ r_c = f_c a_{\text{B}}. \]

(38)

\( f_c \) must be comparable to \( c_3 \), but is expected to be slightly larger than \( c_3 \) as defined in Eq. (28) (since touching spheres leave empty space). Actually the average value of \( f_c \) over all the elemental solids is 3 instead of 2.6 for \( c_3 \).

From dimensional considerations, any quantity \( A \) pertaining to the solid state is expected to be proportional to a combination \( \hbar^2 m_a^2 a_{\text{B}}^2 = A_0 \) such that the dimensions of \( A_0 \) to be the same as those of \( A : [A_0] = [A] \). Of course, \( A \) may depend on other quantities as well (e.g. the type of atom as determined by the atomic number \( Z \), the atomic mass \( m_a \), the velocity of light \( c \), the temperature \( T \), the pressure \( P \), etc.). All these dependences must enter in a dimensionless form by dividing by the corresponding units. Finally, we expect to improve the numerical estimates by replacing \( a_{\text{B}} \) by \( r_c = f_c a_{\text{B}} \) where \( 1 \leq f_c \leq 6 \) (this replacement is denoted by a prime). Thus we have the following general formula

\[ A = A'_0 F \left( Z, m_a / m_c, T / T'_c, P / P'_c, c / c'_c, \ldots \right), \]

(39)
where the function $F$ cannot be determined by dimensional arguments. Physical considerations or a full theory is needed. In what follows we shall apply Eq. (39) to several basic quantities characterizing the solid state. (The dependence on $m_a/m_* T/T'_* \ldots$ will be included only when physical arguments make it necessary). Thus we have:

Cohesive energy per atom $\varepsilon_c$:

$$
\varepsilon_c = \eta_1 \frac{\hbar^2}{m_r c^2} = \eta_1 \frac{27.2}{f_c^2} \text{eV/atom} = 625 \text{kcal/mol}, \quad \eta_1 \approx 1,
$$

(40)

Bulk modulus, $B$:

$$
B = \eta_2 \frac{\hbar^2}{m_r c^2} = \eta_2 \frac{294}{f_c^2} \times 10^{11} \text{N/m}^2, \quad \eta_2 \approx 0.6,
$$

(41)

Velocity of sound $v_s$:

$$
v_s = \eta_3 \frac{\hbar}{m_r c} \sqrt{\frac{m_e}{m_a}} \approx \frac{82}{f_c \sqrt{A_w}} \text{km/s}, \quad \eta_3 \approx 1.6.
$$

(42)

In $v_s$ the atomic mass $m_a$ must enter as $1/\sqrt{m_a}$, since sound in condensed matter propagates through migrating vibrations of the atoms; $\eta_1$ for hydrogen or Van Der Waals bonds is much smaller.

Shear modulus $G$:

$$
G = \eta_4 B; \quad 0.1 \leq \eta_4 \leq 1.
$$

(43)

Critical shear for plastic deformation, $\tau_c$:

$$
\tau_c \approx \frac{G}{100}.
$$

(44)

Eq. (44) is a very rough average kind of approximation. It is difficult to estimate $\tau_c$ because it depends not only on microscopic quantities but on mesoscopic parameters as well, such as the existence of microcracks, and the concentration, mobility, and rate of creation of dislocations.

Estimate of the highest frequency of lattice oscillation. In analogy with Eq. (34) we obtain

$$
\omega_{\text{max}} = \eta_5 \frac{e^2}{\hbar r_c} \sqrt{\frac{m_e}{m_a}} = \eta_5 \frac{96.8}{f_c \sqrt{A_w}} \times 10^{13} \text{rad/s}, \quad \eta_5 \approx 1.
$$

(45)

The Debye temperature $\Theta_D = h\omega_{\text{max}} / k_B$ is approximately

$$
\Theta_D = \eta_6 \frac{7390}{f_c \sqrt{A_w}} \text{K}, \quad \eta_6 \approx 1.
$$

(46)
TABLE 3. Comparison of the results of Eqs. (40), (41), (42), and (46) with the experimental data [2] for four of the most important solids (technologically and historically).

<table>
<thead>
<tr>
<th></th>
<th>Fe</th>
<th>Al</th>
<th>Cu</th>
<th>Si</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>Exp</td>
<td>Estimate</td>
<td>Exp</td>
</tr>
<tr>
<td>$A_e$</td>
<td>55.85</td>
<td>26.98</td>
<td>63.55</td>
<td>28.09</td>
</tr>
<tr>
<td>$f_f$ (eV/atom)</td>
<td>3.73</td>
<td>4.28</td>
<td>3.04</td>
<td>3.39</td>
</tr>
<tr>
<td>$B$ (10^3 N/m^2)</td>
<td>1.29</td>
<td>1.68</td>
<td>0.73</td>
<td>0.72</td>
</tr>
<tr>
<td>$v_s$ (km/s)</td>
<td>4.06</td>
<td>4.63</td>
<td>5.28</td>
<td>5.68</td>
</tr>
<tr>
<td>$\Theta_B$ (K)</td>
<td>366</td>
<td>464</td>
<td>476</td>
<td>426</td>
</tr>
</tbody>
</table>

Liquids: Velocity of sea waves

This velocity must depend on the acceleration of gravity, $g$, which tends to restore equilibrium, possibly on the wavelength $\lambda$, and on the depth of the sea $d$ assumed constant (the density $\rho_M$ of the water does not enter, since $g$ is the same for all types of liquids). Thus we have

$$v^2 = (g / k) F(kd),$$

(47)

where dimensional analysis cannot determine the function $F$ of the dimensional quantity $kd$. However, it is physically clear that in the limit $kd \gg 1$, the depth of the sea plays no role. Similarly, in the opposite limit of $kd \ll 1$ the depth of the sea will dominate.

$$F(kd) \rightarrow \text{constant, } kd \gg 1,$$

(48a)

$$F(kd) \rightarrow \text{constant} \times kd, \quad kd \ll 1.$$

(48b)

It turns out that the constant in both cases is equal to one. Thus $F(x)$ starts as $x$ (for $x \rightarrow 0$) and ends up as 1 for $x \rightarrow \infty$ (actually, $F(x) = \tanh(x)$ [4]); therefore,

$$v = \sqrt{gd}, \quad \text{if } \lambda \gg d,$$

(49a)

$$v = \sqrt{g\lambda / 2\pi}, \quad \text{if } 1 \text{cm} \ll \lambda \ll d.$$

(49b)

The reason for the inequality $\lambda \gg 1$ cm is the following: For wavelengths $\lambda = 1$ cm, or smaller, there is another energy per unit mass (besides the gravitational one) which tends to restore equilibrium; this is the surface tension which depends on the coefficient of surface tension $\sigma$, on the wavelength (or preferably on $k$) and on the density, $\rho_M$. Since these two potential energies per unit mass are additive, they would make two additive contributions to the square of the velocity:

$$v^2 = v_g^2 + v_\sigma^2,$$

(50)

where $v_g^2$ is given by (47) and, by dimensional considerations, $v_\sigma^2 = \eta_\sigma \sigma k / \rho_M$ (where it turns out that $\eta_\sigma = 1$).

\[\text{instead of }\]

Instead of $\lambda$ it is preferable to use the wavenumber $k = 2\pi / \lambda$, since it is $\sin(\lambda x)$ that determines the spatial variation of a wave.
FIGURE 2. Sea wave velocity vs. wavelength. Very short wavelength waves are controlled by surface tension. Very long wavelength waves, as in tsunamis, propagate very fast; e.g. for $d = 2.5 \text{ km}$, and $\lambda \gg d$, $v = 158 \text{ m/s} = 569 \text{ km/h}$.

The surface tension coefficient can be estimated by arguments similar to the ones used in estimating the difference between nucleons on the surface and nucleons on the bulk. If we choose the reasonable value of 0.45 eV/molecule for the water molecules held together by the weak hydrogen bond we find for $\sigma = 0.11 \text{ J/m}^2$ vs. the experimental value of 0.073 J/m$^2$.

Thus the velocity vs. wavelength for sea waves is as in Fig.2. Since, wind can excite sea waves more efficiently when its velocity $v_w$ coincides with the sea waves velocity $v$, we expect short wavelength waves ($\lambda = 1.7 \text{ cm}$) to be the first to be excited as the wind picks up speed and reaches the critical value of about 0.8 km/h.

**Planets: How high a mountain can be?**

A planet is almost spherical which means that the height $h_{\text{max}}$ of the highest mountain ought to be much smaller than the radius $R$ of the planet. What forces a planet to be spherical, is, of course, the gravitational energy. This implies that in a planet the gravitational energy must be at least comparable to the electrostatic one. Hence, its size must be such that

$$E_G = \frac{3}{5} \frac{GM^2}{R} = 2N_a e_h = 2N_a \frac{h^2}{m_n a_n^2 f_c^3} = E_{\text{elect}}.$$  \hspace{1cm} (51)

Taking into account that $M = N_a m_n$ ($m_n = 1823 m_e$) and $(4\pi/3)R^3 = N_a (4\pi/3)a_n^3 f_c^3$, we obtain

$$AN_a = \left(\frac{10}{3 f_c}\right)^{3/2} \frac{1}{A^2} \left(\frac{\alpha}{\alpha_e}\right)^{3/2},$$  \hspace{1cm} (52)

where the ratio of the electric to the gravitational strength $\alpha/\alpha_e = e^2/Gm_e^2 = 1.24 \times 10^{36}$ (see Table 2). If we choose $f_c = 2.3^{110}$ and we use Eq. (37) with the mean density of Earth 5.515 g/cm$^3$ we find the number of nucleons $AN_a$ in a planet where the gravitational energy is about equal to the electrostatic energy:

---

\(^{110}\) Because of the gravitational compression the quantity $f_c$ is expected to be smaller than in an ordinary solid.
\[
A N_a = 3.81 \times 10^{51}.
\]

The corresponding radius is

\[
R = 6.46 \times 10^6 \text{ m},
\]

while for Earth \( A N_e = 3.59 \times 10^{51} \) and \( R = 6.38 \times 10^6 \text{ m} \).

The highest possible mountain in a rocky planet occurs when the shear stress \( \tau \) at its base is equal to the critical shear stress \( \tau_c \) for plastic flow (see Eq. (44)). Combining Eqs. (41), (43), and (44) we have for \( \tau_c \)

\[
\tau_c = \frac{1}{1000} \frac{h^2}{m a_b f_c^3}.
\]

The shear stress is equal to \( \gamma W / S \), where \( \gamma = 2/3 \), \( W \) is the weight of the mountain and \( S \) is the area of its cross-section at its base. If \( V = \frac{1}{3} Sh \) is the volume of the mountain and \( h \) its height, the weight is \( W = g V \rho_m \), where \( \rho_m \) is given by Eq. (37) and \( g \) is given \( GM / R^2 \); \( M = (4\pi / 3) R^3 \rho'_m \). The densities \( \rho_m \) and \( \rho'_m \) are for the mountain and the planet as a whole respectively. Using Eq. (37) and taking for \( \rho_m \) and \( \rho'_m \) the values of Earth \( (3 \text{ g/cm}^3 \) and \( 5.515 \text{ g/cm}^3 \) respectively) we obtain the following result (by choosing \( f_c = 3 \))

\[
\frac{Rh}{a_b^2} = \frac{4\pi}{10^3 \gamma} \left( \frac{2.68}{\rho_f} \right)^2 \frac{\alpha}{\alpha_G} = 3.37 \times 10^{-5} \left( \frac{\alpha}{\alpha_G} \right)
\]

or

\[
R h \approx 1.17 \times 10^{11} \text{ m}^2.
\]

For Earth with \( R = 6.366 \times 10^6 \text{ m} \) the highest possible mountain could not exceed \( h = 18 \text{ km} \), while for Mars with \( R = 3.39 \times 10^6 \text{ m} \), \( h = 34.5 \text{ km} \). The largest asteroid with irregular shape must have \( h = R \); hence its linear dimension is smaller than or about equal to \( 2[1.17 \times 10^{11}]^{1/2} = 700 \text{ km} \).

**Stars, alive or dead**

Consider a spherical interstellar cloud of a given mass \( M = A N_e m_p \) and of radius \( R(t) \) condensing slowly (so that thermodynamic equilibrium to be achieved for each value of \( R \)) to form eventually a star. From virial theorem, the total kinetic energy of the particles making up the cloud is half the gravitational potential energy (in absolute value)

\[
E_k = \frac{1}{2} \frac{G M^2}{R}.
\]

The kinetic energy has a quantum mechanical contribution (see Eq. (2a)) and a thermal contribution. We have by a reasonable interpolation the following expression:

\[
E_k = \left[ \left( 2.87 h^2 N_e^{5/3} / m V^{2/3} \right)^2 + \left( 3 / 2 N_e k_B T \right)^2 \right]^{1/2}.
\]
By combining (55) and (56) we can find the temperature of the cloud vs. the radius (assuming that thermonuclear reactions are not happening). This dependence exhibits a maximum temperature \( T_{\text{max}} \). For the cloud to become a star, \( T_{\text{max}} \) must be clearly larger than the ignition temperature \( T_{\text{ign}} \) for thermonuclear fusion reactions, e.g.:

\[
T_{\text{max}} \approx 2T_{\text{ign}}.
\] (57)

The ignition temperature can be estimated by dimensional consideration taking into account that the fusion occurs by quantum mechanical tunneling of a pair of protons through their Coulomb barrier. Thus \( k_b T_{\text{ign}} \) must depend on \( h \), \( m_p \), \( e \). The result is

\[
k_b T_{\text{ign}} = \eta_b \frac{e^4 m_p}{\hbar^2}; \quad \eta_b = 0.1.
\] (58)

By combining Eqs. (55) to (58) we find the minimum numbers of nucleons \( N_{\nu,\text{min}} \) for an interstellar cloud to become a star.

\[
N_{\nu,\text{min}} \approx 0.5 \left( \frac{m_p}{m_e} \right)^{3/4} \left( \frac{\alpha}{\alpha_G} \right)^{3/2} = 1.93 \times 10^{56}.
\] (59)

If the number of nucleons in a star is very high its average temperature is very high and the pressure of photons (which is proportional to \( T^4 \), see Eqs (6) and (9)) dominates over the pressure of matter. Under these conditions the squeezing gravitational pressure is not enough to compensate the tremendous photonic pressure and the star ceases to exist. This occurs when the number of nucleons exceeds a critical number \( N_{\nu,\text{max}} \) given by

\[
N_{\nu,\text{max}} \approx \left( \frac{\alpha}{\alpha_G} \right)^{3/2} = 1.3 \times 10^{59},
\] (60)

where \( \alpha \) is the dimensionless strength of the strong interaction. (The number of nucleons in our Sun is \( 1.2 \times 10^{57} \), i.e., about 6 times larger than the minimum).

After the thermonuclear fuel is exhausted the star suffers a more or less violent death. At the end of these transient events, gravity keeps squeezing the leftover mass \( M_o \). If this mass \( M_o \) is larger than about three times the mass of our Sun, \( M_s \), there is no mechanism to stop the gravity and the dead star becomes a black hole; the radius \( R_s \) of its horizon can be found from dimensional analysis

\[
R_s = \eta \frac{G M_o}{c^2},
\] (61)

where \( \eta \) turns out to be 2. \( R_s = 14 \text{ km} \) for \( M_o = 5M_s \). Hawking proposed that just outside the horizon of a black hole the gravitational field is so strong that it can absorb one of the partners of the particle/antiparticle pairs created by quantum fluctuations of the vacuum; the other partner cannot be annihilated anymore and it can give rise to radiation. The easiest pair to undergo this complex quantum gravitational process is a pair of photons (since photons have zero rest mass). Thus a black-hole emits EM radiation which turns out to be of the black body type; its effective temperature \( T \) can be found by dimensional considerations, since it must depend on \( \hbar \), on \( GM_o \) and on \( c \)

\[
k_b T = \eta_0 \frac{\hbar c}{R_s} = \eta \frac{\hbar c^3}{2GM_o},
\] (62)
where \( \eta_0 \) turns out to be \( 1/4\pi \). The emission of radiation by the black hole means continuous loss of its energy: \( dU/dt = c^2 dM/dt = -4\pi R^2 \sigma T^4 \). Substituting \( k_b T \) from (62) we obtain for the mass \( M(t) \) of the black hole the following dependence on time:

\[
M^3 = M_0^3 - \frac{\pi^2}{10(8\pi)^2} \frac{\hbar c^4}{G^2} t,
\]

and, hence, a finite lifetime equal to \( t_o = 2.63 \times 10^{-13} M_0^3 \) years, where \( M_o \) is in kg. Having the energy and the temperature, we can obtain the entropy \( S \) of the black hole from the equation \( dU = T dS \). The result is

\[
\frac{S}{k_B} = \frac{A}{4\ell_p^2},
\]

where \( A = 4\pi R^2 \) is the area of the black hole and \( \ell_p \) is the Planck length given by \( \ell_p = \sqrt{\hbar G/c^3} = 1.62 \times 10^{-35} \) m.

Let us consider now the case of \( M_o \) being smaller or about equal the mass of the Sun. In this case the gravity will squeeze the mass \( M_o \) until all electrons are detached form the nuclei and form a cold plasma of \( N_o \) nuclei each of charge \( Z \) (on the average) and \( Z N_o \) free electrons. The energy of this dead star, known as white (or dark) dwarf, consists of the kinetic energy of the electrons (given by Eq. (2a), assuming for the time being that the velocities of electrons are not relativistic) and the gravitational energy (the electrostatic energy is negligible in comparison with the gravitational one). By minimizing the energy with respect to the volume \( V = (4\pi/3)R^3 \) (or with respect to the radius \( R \)) we find

\[
V = \frac{4\pi}{3} R^3 \approx 385 \left( \frac{Z}{A} \right)^5 \frac{\hbar^6}{G^3 m_e^5} \frac{1}{M_o}.
\]

For \( M_o = M_\odot \), the resulting volume is about equal to the volume of the Earth. From Eqs (65) and (2a) we can find the average kinetic energy per electron, \( \epsilon_k \), which is proportional to the \( 4/3 \) power of mass: \( \epsilon_k \sim M_o^{4/3} \). As long as \( \epsilon_k \ll m_e c^2 \), Eq. (65) is valid. However, as the mass \( M \) increases, \( \epsilon_k \) may become larger than \( m_e c^2 \), and Eq. (2b) is the appropriate one for the kinetic energy. In this case both \( E_G \) and \( E_k \) are of the form \( 1/R \). If the coefficient of \( E_G \) is larger than the coefficient of \( E_k \), the electronic quantum kinetic energy cannot stop the gravitational collapse. The equality of the two coefficients occurs when

\[
N_n = 0.77 \left( \frac{1}{\alpha_G} \right)^{3/2} = 1.45 N_e.
\]

The collapse of the white dwarf forces the electrons on the nuclei and the system becomes what is known as neutron star. As long as the neutrons are non-relativistic, they can stop the gravitational collapse at a radius of about 15 km and a density comparable to that of nucleus. However, for larger mass the neutrons too will become relativistic and the total energy will have the form

\[
E = E_k + E_G = \frac{2.32 \beta c \hbar N_e^{4/3}}{V^{1/3}} - \frac{1.612 a G N_e^2 m_u^2}{V^{1/3}},
\]

\(^{10}\) The numerical factor has been changed in order to take into account that the density is not uniform.
where \( a \) and \( \beta \) are numerical factors close to one which depend on the density distribution within the neutron star. Hence the neutron star would collapse to a black hole when its mass exceeds a critical value given by

\[
N_{\nu, \text{crit}} = \left( \frac{2.32 \beta}{1.612 a} \right)^{1/2} \left( \frac{1}{\alpha_G} \right)^{3/2} \approx 3.25 N_k.
\]  

(68)

The last value is obtained by taking \( \beta = a \).

### Universe

We shall assume that the Universe is homogeneous and isotropic (for scales larger than about 100 million light years). Furthermore, we shall use the observation that the universe is expanding at a velocity \( \frac{dR}{dt} = \ddot{R} \) which is proportional to the distance \( R \) of the observed point. The proportionality constant, called Hubble constant and denoted by \( H \) is independent of \( R \) but it does depend on the time \( t \). Besides universal constants, such as \( G \), the main parameter which characterizes the universe is its mass density \( \rho \) or the energy density \( \varepsilon = \rho c^2 \). From dimensional considerations \( H \) is expected to be proportional to \( \sqrt{G \rho} \). This dependence is confirmed by the General Theory of Relativity (GTR); it can also be deduced from Newtonian Mechanics by considering the gravitational force \( F \) on a mass \( m \) at a distance \( R \) from the center of the coordinate system. This force is \( -GM \delta m / R^2 \) where \( M \) is the mass within the volume of the sphere of radius \( R \). By integrating Newton’s equation \( \delta m \ddot{u} = F \) (under \( M \) constant) we obtain.

\[
\left( \frac{\ddot{R}}{R} \right)^2 \equiv H^2 = \frac{8\pi}{3} G \rho - \frac{C}{R^2}.
\]

(69)

\( C \) is the integration constant. Eq. (69) coincides with the final result of the GTR. The argument of inflation leads to the conclusion that the last term is absent. Setting \( C = 0 \) implies, according to GTR, that the geometry of space (at large scales) is euclidean. Usually equation (69) is written with \( C = 0 \) and \( \rho \) replaced by \( \varepsilon / c^2 \):

\[
\left( \frac{\ddot{R}}{R} \right)^2 = \frac{8\pi G}{3c^2} \varepsilon.
\]

(70)

To solve Eq. (70) we have to know the various contributions to \( \varepsilon \) and their dependence on \( R \). One contribution is from protons, neutrons, etc; it is called baryonic contribution and its dependence on \( R \) is \( \varepsilon_{\text{b}} \propto 1/R^3 \), since their number is conserved. Another is from neutrinos which is also \( 1/R^3 \) (for \( t \) not so close to the big bang time). A third one comes from photons, which according to Eq. (7) is proportional to \( T^4 \sim 1/R^4 \) (since the photon entropy \( S \sim R^3 T^4 \) is conserved). Several recent observations strongly indicate that there are two additional contributions: One is associated to the so called dark matter (which is believed to consist of exotic particles (not observed experimentally) which do not couple to the EM field); \( \varepsilon_{\text{DM}} \) is also expected to depend on \( R \) as \( 1/R^3 \); the other, called dark energy, seems to be independent of \( R \) being similar in this respect to the cosmological constant introduced (and later redrawn) by Einstein.

At the early history of the Universe the photonic contribution to \( \varepsilon \) dominated because of its \( 1/R^4 \) dependence. Then, in this eposque, Eq. (70) becomes \( \dot{R}/R \sim 1/R^2 \) or \( RR = \text{const} \) or

\[
R \sim t^{1/2}, \quad \text{photon domination.}
\]

(71)

At a later period the matter plus dark energy contributions dominate, which plugged in (70) give,
\[ \dot{R} = \left[ \frac{c_1}{R} + c_2 R^2 \right]^{1/2}. \quad (72) \]

Eq. (72) can be solved analytically to give

\[ R(t) = \left( \frac{c_1}{c_2} \right)^{1/3} \left[ \sinh \left( \frac{3}{2} \sqrt{c_2} t \right) \right]^{2/3}, \quad (73) \]

which for relatively small argument gives

\[ R \sim t^{2/3}, \quad \text{matter domination}, \quad (74) \]

while for very large times

\[ R \sim e^{H_0 t}, \quad H_0 = \sqrt{c_2}, \quad \text{dark energy domination}. \quad (75) \]

By introducing the present day value of \( R_0 \) and the present value of the Hubble constant, we obtain from (73) the following relation

\[ R(t) = 0.7 R_0 \left[ \sinh \left( 0.09 t \right) \right]^{2/3}, \quad t \approx 2 \times 10^5 \text{ years}, \quad (76) \]

where \( t \) is in billion years. Eq. (76) suggests that the second derivative of \( R \) with respect to time from negative becomes positive at \( t = 7.5 \) billion years after the big bang. Since the present time is 13.7 billion years after the big bang, we can reach the conclusion that we live in the period of accelerated expansion of the Universe. Data from the satellite WMAP [5,6] and data from distant supernovae show that this is actually the case.

**REFERENCES**